

# An algorithm for the classification of smooth Fano polytopes

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## Abstract

We present an algorithm that produces the classification list of smooth Fano  $d$ -polytopes for any given  $d \geq 1$ . The input of the algorithm is a single number, namely the positive integer  $d$ . The algorithm has been used to classify smooth Fano  $d$ -polytopes for  $d \leq 7$ . There are 7622 isomorphism classes of smooth Fano 6-polytopes and 72256 isomorphism classes of smooth Fano 7-polytopes.

## 1 Introduction

Isomorphism classes of smooth toric Fano varieties of dimension  $d$  correspond to isomorphism classes of so-called smooth Fano  $d$ -polytopes, which are fully dimensional convex lattice polytopes in  $\mathbb{R}^d$ , such that the origin is in the interior of the polytopes and the vertices of every facet is a basis of the integral lattice  $\mathbb{Z}^d \subset \mathbb{R}^d$ . Smooth Fano  $d$ -polytopes have been intensively studied for the last decades. They have been completely classified up to isomorphism for  $d \leq 4$  ([1], [18], [3], [15]). Under additional assumptions there are classification results valid in every dimension.

To our knowledge smooth Fano  $d$ -polytopes have been classified in the following cases:

- When the number of vertices is  $d + 1$ ,  $d + 2$  or  $d + 3$  ([9],[2]).
- When the number of vertices is  $3d$ , which turns out to be the upper bound on the number of vertices ([6]).
- When the number of vertices is  $3d - 1$  ([19]).
- When the polytopes are centrally symmetric ([17]).
- When the polytopes are pseudo-symmetric, i.e. there is a facet  $F$ , such that  $-F$  is also a facet ([8]).
- When there are many pairs of centrally symmetric vertices ([5]).

- When the corresponding toric  $d$ -folds are equipped with an extremal contraction, which contracts a toric divisor to a point ([4]) or a curve ([16]).

Recently a complete classification of smooth Fano 5-polytopes has been announced ([12]). The approach is to recover smooth Fano  $d$ -polytopes from their image under the projection along a vertex. This image is a *reflexive*  $(d-1)$ -polytope (see [3]), which is a fully-dimensional lattice polytope containing the origin in the interior, such that the dual polytope is also a lattice polytope. Reflexive polytopes have been classified up to dimension 4 using the computer program PALP ([10],[11]). Using this classification and PALP the authors of [12] succeed in classifying smooth Fano 5-polytopes.

In this paper we present an algorithm that classifies smooth Fano  $d$ -polytopes for any given  $d \geq 1$ . We call this algorithm SFP (for Smooth Fano Polytopes). The input is the positive integer  $d$ , nothing else is needed. The algorithm has been implemented in C++, and used to classify smooth Fano  $d$ -polytopes for  $d \leq 7$ . For  $d = 6$  and  $d = 7$  our results are new:

**Theorem 1.1.** *There are 7622 isomorphism classes of smooth Fano 6-polytopes and 72256 isomorphism classes of smooth Fano 7-polytopes.*

The classification lists of smooth Fano  $d$ -polytopes,  $d \leq 7$ , are available on the authors homepage: <http://home.imf.au.dk/oebro>

A key idea in the algorithm is the notion of a special facet of a smooth Fano  $d$ -polytope (defined in section 3.1): A facet  $F$  of a smooth Fano  $d$ -polytope is called *special*, if the sum of the vertices of the polytope is a non-negative linear combination of vertices of  $F$ . This allows us to identify a finite subset  $\mathcal{W}_d$  of the lattice  $\mathbb{Z}^d$ , such that any smooth Fano  $d$ -polytope is isomorphic to one whose vertices are contained in  $\mathcal{W}_d$  (theorem 3.6). Thus the problem of classifying smooth Fano  $d$ -polytopes is reduced to the problem of considering certain subsets of  $\mathcal{W}_d$ .

We then define a total order on finite subsets of  $\mathbb{Z}^d$  and use this to define a total order on the set of smooth Fano  $d$ -polytopes, which respects isomorphism (section 4). The SFP-algorithm (described in section 5) goes through certain finite subsets of  $\mathcal{W}_d$  in increasing order, and outputs smooth Fano  $d$ -polytopes in increasing order, such that any smooth Fano  $d$ -polytope is isomorphic to exactly one in the output list.

As a consequence of the total order on smooth Fano  $d$ -polytopes, the algorithm needs not consult the previous output to check for isomorphism to decide whether or not to output a constructed polytope.

## 2 Smooth Fano polytopes

We fix a notation and prove some simple facts about smooth Fano polytopes.

The convex hull of a set  $K \in \mathbb{R}^d$  is denoted by  $\text{conv}K$ . A *polytope* is the convex hull of finitely many points. The dimension of a polytope  $P$  is the dimension of the affine hull,  $\text{aff}P$ , of the polytope  $P$ . A *k-polytope* is a polytope of dimension  $k$ . A *face* of a polytope is the intersection of a supporting hyperplane with the polytope. Faces of polytopes are polytopes. Faces of dimension 0 are called *vertices*, while faces of codimension 1 and 2 are called *facets* and *ridges*, respectively. The set of vertices of a polytope  $P$  is denoted by  $\mathcal{V}(P)$ .

**Definition 2.1.** A convex lattice polytope  $P$  in  $\mathbb{R}^d$  is called a smooth Fano  $d$ -polytope, if the origin is contained in the interior of  $P$  and the vertices of every facet of  $P$  is a  $\mathbb{Z}$ -basis of the lattice  $\mathbb{Z}^d \subset \mathbb{R}^d$ .

We consider two smooth Fano  $d$ -polytopes  $P_1, P_2$  to be *isomorphic*, if there exists a bijective linear map  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , such that  $\varphi(\mathbb{Z}^d) = \mathbb{Z}^d$  and  $\varphi(P_1) = P_2$ .

Whenever  $F$  is a  $(d-1)$ -simplex in  $\mathbb{R}^d$ , such that  $0 \notin \text{aff}F$ , we let  $u_F \in (\mathbb{R}^d)^*$  be the unique element determined by  $\langle u_F, F \rangle = \{1\}$ . For every  $w \in \mathcal{V}(F)$  we define  $u_F^w \in (\mathbb{R}^d)^*$  to be the element where  $\langle u_F^w, w \rangle = 1$  and  $\langle u_F^w, w' \rangle = 0$  for every  $w' \in \mathcal{V}(F)$ ,  $w' \neq w$ . Then  $\{u_F^w | w \in \mathcal{V}(F)\}$  is the basis of  $(\mathbb{R}^d)^*$  dual to the basis  $\mathcal{V}(F)$  of  $\mathbb{R}^d$ .

When  $F$  is a facet of a smooth Fano polytope and  $v \in \mathcal{V}(P)$ , we certainly have  $\langle u_F, v \rangle \in \mathbb{Z}$  and

$$\langle u_F, v \rangle = 1 \iff v \in \mathcal{V}(F) \quad \text{and} \quad \langle u_F, v \rangle \leq 0 \iff v \notin \mathcal{V}(F).$$

The lemma below concerns the relation between the elements  $u_F$  and  $u_{F'}$ , when  $F$  and  $F'$  are adjacent facets.

**Lemma 2.2.** Let  $F$  be a facet of a smooth Fano polytope  $P$  and  $v \in \mathcal{V}(F)$ . Let  $F'$  be the unique facet which intersects  $F$  in a ridge  $R$  of  $P$ ,  $v \notin \mathcal{V}(R)$ . Let  $v' = \mathcal{V}(F') \setminus \mathcal{V}(R)$ .

Then

1.  $\langle u_F^v, v' \rangle = -1$ .
2.  $\langle u_F, v' \rangle = \langle u_{F'}, v \rangle$ .
3.  $\langle u_{F'}, x \rangle = \langle u_F, x \rangle + \langle u_F^v, x \rangle (\langle u_F, v' \rangle - 1)$  for any  $x \in \mathbb{R}^d$ .
4. In particular,

- $\langle u_F^v, x \rangle < 0$  iff  $\langle u_{F'}, x \rangle > \langle u_F, x \rangle$ .
- $\langle u_F^v, x \rangle > 0$  iff  $\langle u_{F'}, x \rangle < \langle u_F, x \rangle$ .
- $\langle u_F^v, x \rangle = 0$  iff  $\langle u_{F'}, x \rangle = \langle u_F, x \rangle$ .

for any  $x \in \mathbb{R}^d$ .

5. Suppose  $x \neq v'$  is a vertex of  $P$  where  $\langle u_F^v, x \rangle < 0$ . Then  $\langle u_F, v' \rangle > \langle u_F, x \rangle$ .

*Proof.* The sets  $\mathcal{V}(F)$  and  $\mathcal{V}(F')$  are both bases of the lattice  $\mathbb{Z}^d$  and the first statement follows.

We have  $v + v' \in \text{span}(F \cap F')$ , and then the second statement follows.

Use the previous statements to calculate  $\langle u_{F'}, x \rangle$ .

$$\begin{aligned} \langle u_{F'}, x \rangle &= \langle u_{F'}, \sum_{w \in \mathcal{V}(F)} \langle u_F^w, x \rangle w \rangle \\ &= \sum_{w \in \mathcal{V}(F) \setminus \{v\}} \langle u_F^w, x \rangle + \langle u_F^v, x \rangle \langle u_{F'}, v \rangle \\ &= \langle u_F, x \rangle + \langle u_F^v, x \rangle (\langle u_{F'}, v \rangle - 1) \\ &= \langle u_F, x \rangle + \langle u_F^v, x \rangle (\langle u_F, v' \rangle - 1). \end{aligned}$$

As  $\langle u_F, v' \rangle - 1 < 0$  the three equivalences follow directly.

Suppose there is a vertex  $x \in \mathcal{V}(P)$ , such that  $\langle u_F^v, x \rangle < 0$  and  $\langle u_F, v' \rangle \leq \langle u_F, x \rangle$ . Then

$$\langle u_{F'}, x \rangle = \langle u_F, x \rangle + \langle u_F^v, x \rangle (\langle u_F, v' \rangle - 1) \geq \langle u_F, x \rangle - (\langle u_F, v' \rangle - 1) \geq 1.$$

Hence  $x$  is on the facet  $F'$ . But this cannot be the case as  $\mathcal{V}(F') = \{v'\} \cup \mathcal{V}(F) \setminus \{v\}$ . Thus no such  $x$  exists.

And we're done.  $\square$

In the next lemma we show a lower bound on the numbers  $\langle u_F^w, v \rangle$ ,  $w \in \mathcal{V}(F)$ , for any facet  $F$  and any vertex  $v$  of a smooth Fano  $d$ -polytope.

**Lemma 2.3.** *Let  $F$  be a facet and  $v$  a vertex of a smooth Fano polytope  $P$ . Then*

$$\langle u_F^w, v \rangle \geq \begin{cases} 0 & \langle u_F, v \rangle = 1 \\ -1 & \langle u_F, v \rangle = 0 \\ \langle u_F, v \rangle & \langle u_F, v \rangle < 0 \end{cases}$$

for every  $w \in \mathcal{V}(F)$ .

*Proof.* When  $\langle u_F, v \rangle = 1$  the statement is obvious.

Suppose  $\langle u_F, v \rangle = 0$  and  $\langle u_F^w, v \rangle < 0$  for some  $w \in \mathcal{V}(F)$ . Let  $F'$  be the unique facet intersecting  $F$  in the ridge  $\text{conv}\{\mathcal{V}(F) \setminus \{w\}\}$ . By lemma 2.2  $\langle u_{F'}, v \rangle > 0$ . As  $\langle u_{F'}, v \rangle \in \mathbb{Z}$  we must have  $\langle u_{F'}, v \rangle = 1$ . This implies  $\langle u_F, v \rangle = -1$ .

Suppose  $\langle u_F, v \rangle < 0$  and  $\langle u_F^w, v \rangle < \langle u_F, v \rangle \leq -1$  for some  $w \in \mathcal{V}(F)$ . Let  $F' \neq F$  be the facet containing the ridge  $\text{conv}\{\mathcal{V}(F) \setminus \{w\}\}$ , and let  $w'$  be the unique vertex in  $\mathcal{V}(F') \setminus \mathcal{V}(F)$ . Then by lemma 2.2

$$\langle u_{F'}, v \rangle = \langle u_F, v \rangle + \langle u_F^w, v \rangle (\langle u_F, w' \rangle - 1) \geq \langle u_F, v \rangle - \langle u_F^w, v \rangle.$$

If  $\langle u_F, v \rangle - \langle u_F^w, v \rangle > 0$ , then  $v$  is on the facet  $F'$ . But this is not the case as  $\langle u_F^w, v \rangle < -1$ . We conclude that  $\langle u_F^w, v \rangle \geq \langle u_F, v \rangle$ .  $\square$

When  $F$  is a facet and  $v$  a vertex of a smooth Fano  $d$ -polytope  $P$ , such that  $\langle u_F, v \rangle = 0$ , we can say something about the face lattice of  $P$ .

**Lemma 2.4** ([7] section 2.3 remark 5(2), [13] lemma 5.5). *Let  $F$  be a facet and  $v$  be vertex of a smooth Fano polytope  $P$ . Suppose  $\langle u_F, v \rangle = 0$ . Then  $\text{conv}\{\{v\} \cup \mathcal{V}(F) \setminus \{w\}\}$  is a facet of  $P$  for every  $w \in \mathcal{V}(F)$  with  $\langle u_F^w, v \rangle = -1$ .*

*Proof.* Follows from the proof of lemma 2.3.  $\square$

### 3 Special embeddings of smooth Fano polytopes

In this section we find a concrete finite subset  $\mathcal{W}_d$  of  $\mathbb{Z}^d$  with the nice property that any smooth Fano  $d$ -polytope is isomorphic to one whose vertices are contained in  $\mathcal{W}_d$ . The problem of classifying smooth Fano  $d$ -polytopes is then reduced to considering subsets of  $\mathcal{W}_d$ .

#### 3.1 Special facets

The following definition is a key concept.

**Definition 3.1.** *A facet  $F$  of a smooth Fano  $d$ -polytope  $P$  is called special, if the sum of the vertices of  $P$  is a non-negative linear combination of  $\mathcal{V}(F)$ , that is*

$$\sum_{v \in \mathcal{V}(P)} v = \sum_{w \in \mathcal{V}(F)} a_w w, \quad a_w \geq 0.$$

Clearly, any smooth Fano  $d$ -polytope has at least one special facet.

Let  $F$  be a special facet of a smooth Fano  $d$ -polytope  $P$ . Then

$$0 \leq \langle u_F, \sum_{v \in \mathcal{V}(P)} v \rangle = d + \sum_{v \in \mathcal{V}(P), \langle u_F, v \rangle < 0} \langle u_F, v \rangle,$$

which implies  $-d \leq \langle u_F, v \rangle \leq 1$  for any vertex  $v$  of  $P$ . By using the lower bound on the numbers  $\langle u_F^w, v \rangle$ ,  $w \in \mathcal{V}(F)$  (see lemma 2.3), we can find an explicit finite subset of the lattice  $\mathbb{Z}^d$ , such that every  $v \in \mathcal{V}(P)$  is contained in this subset. In the following lemma we generalize this observation to subsets of  $\mathcal{V}(P)$  containing  $\mathcal{V}(F)$ .

**Lemma 3.2.** *Let  $P$  be a smooth Fano polytope. Let  $F$  be a special facet of  $P$  and let  $V$  be a subset of  $\mathcal{V}(P)$  containing  $\mathcal{V}(F)$ , whose sum is  $\nu$ .*

$$\nu = \sum_{v \in V} v.$$

*Then*

$$\langle u_F, \nu \rangle \geq 0$$

and

$$\langle u_F^w, \nu \rangle \leq \langle u_F, \nu \rangle + 1$$

for every  $w \in \mathcal{V}(F)$ .

*Proof.* For convenience we set  $U = \mathcal{V}(P) \setminus V$  and  $\mu = \sum_{v \in U} v$ . Since  $F$  is a special facet we know that

$$0 \leq \langle u_F, \sum_{v \in \mathcal{V}(P)} v \rangle = \langle u_F, \nu \rangle + \langle u_F, \mu \rangle.$$

The set  $\mathcal{V}(F)$  is contained in  $V$  so  $\langle u_F, v \rangle \leq 0$  for every  $v$  in  $U$ , hence  $\langle u_F, \nu \rangle \geq 0$ .

Suppose that for some  $w \in \mathcal{V}(F)$  we have  $\langle u_F^w, \nu \rangle > \langle u_F, \nu \rangle + 1$ . By lemma 2.3 we know that

$$\langle u_F^w, v \rangle \geq \begin{cases} -1 & \langle u_F, v \rangle = 0 \\ \langle u_F, v \rangle & \langle u_F, v \rangle < 0 \end{cases}$$

for every vertex  $v \in \mathcal{V}(P) \setminus \mathcal{V}(F)$ . There is at most one vertex  $v$  of  $P$ ,  $\langle u_F, v \rangle = 0$ , with negative coefficient  $\langle u_F^w, v \rangle$  (lemma 2.4). So

$$\langle u_F^w, \mu \rangle \geq \langle u_F, \mu \rangle - 1.$$

Now, consider  $\langle u_F^w, \sum_{v \in \mathcal{V}(P)} v \rangle$ .

$$\langle u_F^w, \sum_{v \in \mathcal{V}(P)} v \rangle = \langle u_F^w, \nu \rangle + \langle u_F^w, \mu \rangle > \langle u_F, \nu \rangle + \langle u_F, \mu \rangle = \langle u_F, \sum_{v \in \mathcal{V}(P)} v \rangle.$$

But this implies that  $\langle u_F^x, \sum_{v \in \mathcal{V}(P)} v \rangle$  is negative for some  $x \in \mathcal{V}(F)$ . A contradiction.  $\square$

**Corollary 3.3.** *Let  $F$  be a special facet and  $v$  any vertex of a smooth Fano  $d$ -polytope. Then  $-d \leq \langle u_F, v \rangle \leq 1$  and*

$$\left. \begin{array}{c} 0 \\ -1 \\ \langle u_F, v \rangle \end{array} \right\} \leq \langle u_F^w, v \rangle \leq \left\{ \begin{array}{ll} 1 & , \langle u_F, v \rangle = 1 \\ d-1 & , \langle u_F, v \rangle = 0 \\ d + \langle u_F, v \rangle & , \langle u_F, v \rangle < 0 \end{array} \right.$$

for every  $w \in \mathcal{V}(F)$ .

*Proof.* For  $\langle u_F, v \rangle = 1$  the statement is obvious. When  $\langle u_F, v \rangle = 0$  the coefficients of  $v$  with respect to the basis  $\mathcal{V}(F)$  is bounded below by  $-1$  (lemma 2.3), so no coefficient exceeds  $d-1$ .

So the case  $\langle u_F, v \rangle < 0$  remains. The lower bound is by lemma 2.3. Use lemma 3.2 on the subset  $V = \mathcal{V}(F) \cup \{v\}$  to prove the upper bound.  $\square$

### 3.2 Special embeddings

Let  $(e_1, \dots, e_d)$  be a fixed basis of the lattice  $\mathbb{Z}^d \subset \mathbb{R}^d$ .

**Definition 3.4.** *Let  $P$  be a smooth Fano  $d$ -polytope. Any smooth Fano  $d$ -polytope  $Q$ , with  $\text{conv}\{e_1, \dots, e_d\}$  as a special facet, is called a special embedding of  $P$ , if  $P$  and  $Q$  are isomorphic.*

Obviously, for any smooth Fano polytope  $P$ , there exists at least one special embedding of  $P$ . As any polytope has finitely many facets, there exists only finitely many special embeddings of  $P$ .

Now we define a subset of  $\mathbb{Z}^d$  which will play an important part in what follows.

**Definition 3.5.** *By  $\mathcal{W}_d$  we denote the maximal set (with respect to inclusion) of lattice points in  $\mathbb{Z}^d$  such that*

1. *The origin is not contained in  $\mathcal{W}_d$ .*
2. *The points in  $\mathcal{W}_d$  are primitive lattice points.*
3. *If  $a_1 e_1 + \dots + a_d e_d \in \mathcal{W}_d$ , then  $-d \leq a \leq 1$  for  $a = a_1 + \dots + a_d$  and*

$$\left. \begin{array}{c} 0 \\ -1 \\ a \end{array} \right\} \leq a_i \leq \left\{ \begin{array}{ll} 1 & , a = 1 \\ d-1 & , a = 0 \\ d+a & , a < 0 \end{array} \right.$$

*for every  $i = 1, \dots, d$ .*

The next theorem is one of the key results in this paper. It allows us to classify smooth Fano  $d$ -polytopes by considering subsets of the explicitly given set  $\mathcal{W}_d$ .

**Theorem 3.6.** *Let  $P$  be an arbitrary smooth Fano  $d$ -polytope, and  $Q$  any special embedding of  $P$ . Then  $\mathcal{V}(Q)$  is contained in the set  $\mathcal{W}_d$ .*

*Proof.* Follows directly from corollary 3.3 and the definition of  $\mathcal{W}_d$ . □

## 4 Total ordering of smooth Fano polytopes

In this section we define a total order on the set of smooth Fano  $d$ -polytopes for any fixed  $d \geq 1$ .

Throughout the section  $(e_1, \dots, e_d)$  is a fixed basis of the lattice  $\mathbb{Z}^d$ .

### 4.1 The order of a lattice point

We begin by defining a total order  $\preceq$  on  $\mathbb{Z}^d$ .

**Definition 4.1.** Let  $x = x_1e_1 + \dots + x_de_d$ ,  $y = y_1e_1 + \dots + y_de_d$  be two lattice points in  $\mathbb{Z}^d$ . We define  $x \preceq y$  if and only if

$$(-x_1 - \dots - x_d, x_1, \dots, x_d) \leq_{\text{lex}} (-y_1 - \dots - y_d, y_1, \dots, y_d),$$

where  $\leq_{\text{lex}}$  is the lexicographical ordering on the product of  $d+1$  copies of the ordered set  $(\mathbb{Z}, \leq)$ .

The ordering  $\preceq$  is a total order on  $\mathbb{Z}^d$ .

**Example.**  $(0, 1) \prec (-1, 1) \prec (1, -1) \prec (-1, 0)$ .

Let  $V$  be any nonempty finite subset of lattice points in  $\mathbb{Z}^d$ . We define  $\max V$  to be the maximal element in  $V$  with respect to the ordering  $\preceq$ . Similarly,  $\min V$  is defined to be the minimal element in  $V$ .

A important property of the ordering is shown in the following lemma.

**Lemma 4.2.** Let  $P$  be a smooth Fano  $d$ -polytope, such that  $\text{conv}\{e_1, \dots, e_d\}$  is a facet of  $P$ . For every  $1 \leq i \leq d$ , let  $v_i \neq e_i$  denote the vertex of  $P$ , such that  $\text{conv}\{e_1, \dots, e_{i-1}, v_i, e_{i+1}, \dots, e_d\}$  is a facet of  $P$ .

Then  $v_i = \min\{v \in \mathcal{V}(P) \mid \langle u_F^{e_i}, v \rangle < 0\}$ .

*Proof.* By lemma 2.2.(1) the vertex  $v_i$  is in the set  $\{v \in \mathcal{V}(P) \mid \langle u_F^{e_i}, v \rangle < 0\}$ , and by lemma 2.2.(5) and the definition of the ordering  $\preceq$ ,  $v_i$  is the minimal element in this set.  $\square$

In fact, we have chosen the ordering  $\preceq$  to obtain the property of lemma 4.2, and any other total order on  $\mathbb{Z}^d$  having this property can be used in what follows.

### 4.2 The order of a smooth Fano $d$ -polytope

We can now define an ordering on finite subsets of  $\mathbb{Z}^d$ . The ordering is defined recursively.

**Definition 4.3.** Let  $X$  and  $Y$  be finite subsets of  $\mathbb{Z}^d$ . We define  $X \preceq Y$  if and only if  $X = \emptyset$  or

$$Y \neq \emptyset \wedge (\min X \prec \min Y \vee (\min X = \min Y \wedge X \setminus \{\min X\} \preceq Y \setminus \{\min Y\})).$$

**Example.**  $\emptyset \prec \{(0, 1)\} \prec \{(0, 1), (-1, 1)\} \prec \{(0, 1), (1, -1)\} \prec \{(-1, 1)\}$ .

When  $W$  is a nonempty finite set of subsets of  $\mathbb{Z}^d$ , we define  $\max W$  to be the maximal element in  $W$  with respect to the ordering of subsets  $\preceq$ . Similarly,  $\min W$  is the minimal element in  $W$ .

Now, we are ready to define the order of a smooth Fano  $d$ -polytope.



**Definition 4.4.** Let  $P$  be a smooth Fano  $d$ -polytope. The order of  $P$ ,  $\text{ord}(P)$ , is defined as

$$\text{ord}(P) := \min\{\mathcal{V}(Q) \mid Q \text{ a special embedding of } P\}.$$

The set is non-empty and finite, so  $\text{ord}(P)$  is well-defined.

Let  $P_1$  and  $P_2$  be two smooth Fano  $d$ -polytopes. We say that  $P_1 \leq P_2$  if and only if  $\text{ord}(P_1) \preceq \text{ord}(P_2)$ . This is indeed a total order on the set of isomorphism classes of smooth Fano  $d$ -polytopes.

### 4.3 Permutation of basisvectors and presubsets

The group  $S_d$  of permutations of  $d$  elements acts on  $\mathbb{Z}^d$  in the obvious way by permuting the basisvectors:

$$\sigma.(a_1e_1 + \dots + a_de_d) := a_1e_{\sigma(1)} + \dots + a_de_{\sigma(d)} \quad , \quad \sigma \in S_d.$$

Similarly,  $S_d$  acts on subsets of  $\mathbb{Z}^d$ :

$$\sigma.X := \{\sigma.x \mid x \in X\}.$$

In this notation we clearly have for any special embedding  $P$  of a smooth Fano  $d$ -polytope

$$\text{ord}(P) \preceq \min\{\sigma.\mathcal{V}(P) \mid \sigma \in S_d\}.$$

Let  $V$  and  $W$  be finite subsets of  $\mathbb{Z}^d$ . We say that  $V$  is a *presubset* of  $W$ , if  $V \subseteq W$  and  $v \prec w$  whenever  $v \in V$  and  $w \in W \setminus V$ .

**Example.**  $\{(0, 1), (-1, 1)\}$  is a presubset of  $\{(0, 1), (-1, 1), (1, -1)\}$ , while  $\{(0, 1), (1, -1)\}$  is not.

**Lemma 4.5.** Let  $P$  be a smooth Fano polytope. Then every presubset  $V$  of  $\text{ord}(P)$  is the minimal element in  $\{\sigma.V \mid \sigma \in S_d\}$ .

*Proof.* Let  $\text{ord}(P) = \{v_1, \dots, v_n\}$ ,  $v_1 \prec \dots \prec v_n$ . Suppose there exists a permutation  $\sigma$  and a  $k$ ,  $1 \leq k \leq n$ , such that

$$\sigma.\{v_1, \dots, v_k\} = \{w_1, \dots, w_k\} \prec \{v_1, \dots, v_k\},$$

where  $w_1 \prec \dots \prec w_k$ . Then there is a number  $j$ ,  $1 \leq j \leq k$ , such that  $w_i = v_i$  for every  $1 \leq i < j$  and  $w_j \prec v_j$ .

Let  $\sigma$  act on  $\{v_1, \dots, v_n\}$ .

$$\sigma.\{v_1, \dots, v_n\} = \{x_1, \dots, x_n\} \quad , \quad x_1 \prec \dots \prec x_n.$$

Then  $x_i \preceq v_i$  for every  $1 \leq i < j$  and  $x_j \prec v_j$ . So  $\sigma.\text{ord}(P) \prec \text{ord}(P)$ , but this contradicts the definition of  $\text{ord}(P)$ .  $\square$

## 5 The SFP-algorithm

In this section we describe an algorithm that produces the classification list of smooth Fano  $d$ -polytopes for any given  $d \geq 1$ . The algorithm works by going through certain finite subsets of  $\mathcal{W}_d$  in increasing order (with respect to the ordering defined in the previous section). It will output a subset  $V$  iff  $\text{conv}V$  is a smooth Fano  $d$ -polytope  $P$  and  $\text{ord}(P) = V$ .

Throughout the whole section  $(e_1, \dots, e_d)$  is a fixed basis of  $\mathbb{Z}^d$  and  $I$  denotes the  $(d-1)$ -simplex  $\text{conv}\{e_1, \dots, e_d\}$ .

### 5.1 The SFP-algorithm

The SFP-algorithm consists of three functions,

**SFP**, **AddPoint** and **CheckSubset**.

The finite subsets of  $\mathcal{W}_d$  are constructed by the function **AddPoint**, which takes a subset  $V$ ,  $\{e_1, \dots, e_d\} \subseteq V \subseteq \mathcal{W}_d$ , together with a finite set  $\mathcal{F}$ ,  $I \in \mathcal{F}$ , of  $(d-1)$ -simplices in  $\mathbb{R}^d$  as input. It then goes through every  $v$  in the set

$$\{v \in \mathcal{W}_d \mid \max V \prec v\}$$

in increasing order, and recursively calls itself with input  $V \cup \{v\}$  and some set  $\mathcal{F}'$  of  $(d-1)$ -simplices of  $\mathbb{R}^d$ ,  $\mathcal{F} \subseteq \mathcal{F}'$ . In this way subsets of  $\mathcal{W}_d$  are considered in increasing order.

Whenever **AddPoint** is called, it checks if the input set  $V$  is the vertex set of a special embedding of a smooth Fano  $d$ -polytope  $P$  such that  $\text{ord}(P) = V$ , in which case the polytope  $P = \text{conv}V$  is outputted.

For any given integer  $d \geq 1$  the function **SFP** calls the function **AddPoint** with input  $\{e_1, \dots, e_d\}$  and  $\{I\}$ . In this way a call **SFP**( $d$ ) will make the algorithm go through every finite subset of  $\mathcal{W}_d$  containing  $\{e_1, \dots, e_d\}$ , and smooth Fano  $d$ -polytopes are outputted in strictly increasing order.

It is vital for the effectiveness of the SFP-algorithm, that there is some efficient way to check if a subset  $V \subseteq \mathcal{W}_d$  is a presubset of  $\text{ord}(P)$  for some smooth Fano  $d$ -polytope  $P$ . The function **AddPoint** should perform this check before the recursive call **AddPoint**( $V, \mathcal{F}'$ ).

If  $P$  is any smooth Fano  $d$ -polytope, then any presubset  $V$  of  $\text{ord}(P)$  is the minimal element in the set  $\{\sigma.V \mid \sigma \in S_d\}$  (by lemma 4.5). In other words, if there exists a permutation  $\sigma$  such that  $\sigma.V \prec V$ , then the algorithm should not make the recursive call **AddPoint**( $V$ ).

But this is not the only test we wish to perform on a subset  $V$  before the recursive call. The function **CheckSubset** performs another test: It takes a subset  $V$ ,  $\{e_1, \dots, e_d\} \subseteq V \subseteq \mathcal{W}_d$  as input together with a finite set of  $(d-1)$ -simplices  $\mathcal{F}$ ,  $I \in \mathcal{F}$ , and returns a set  $\mathcal{F}'$  of  $(d-1)$ -simplices containing  $\mathcal{F}$ , if there exists a special embedding  $P$  of a smooth Fano  $d$ -polytope, such that

1.  $V$  is a presubset of  $\mathcal{V}(P)$
2.  $\mathcal{F}$  is a subset of the facets of  $P$

This is proved in theorem 5.1. If no such special embedding exists, then **CheckSubset** returns false in many cases, but not always! Only when **CheckSubset**( $V, \mathcal{F}$ ) returns a set  $\mathcal{F}'$  of simplices, we allow the recursive call **AddPoint**( $V, \mathcal{F}'$ ).

Given input  $V \subseteq \mathcal{W}_d$  and a set  $\mathcal{F}$  of  $(d-1)$ -simplices of  $\mathbb{R}^d$ , the function **CheckSubset** works in the following way: Suppose  $V$  is a presubset of  $\mathcal{V}(P)$  for some special embedding  $P$  of a smooth Fano  $d$ -polytope and  $\mathcal{F}$  is a subset of the facets of  $P$ . Deduce as much as possible of the face lattice of  $P$  and look for contradictions to the lemmas stated in section 2. The more facets we know of  $P$ , the more restrictions we can put on the vertex set  $\mathcal{V}(P)$ , and then on  $V$ . If a contradiction arises, return false. Otherwise, return the deduced set of facets of  $P$ .

The following example illustrates how the function **CheckSubset** works.

## 5.2 An example of the reasoning in CheckSubset

Let  $d = 5$  and  $V = \{v_1, \dots, v_8\}$ , where

$$\begin{aligned} v_1 &= e_1, v_2 = e_2, v_3 = e_3, v_4 = e_4, v_5 = e_5 \\ v_6 &= -e_1 - e_2 + e_4 + e_5, v_7 = e_2 - e_3 - e_4, v_8 = -e_4 - e_5. \end{aligned}$$

Suppose  $P$  is a special embedding of a smooth Fano 5-polytope, such that  $V$  is a presubset of  $\mathcal{V}(P)$ . Certainly, the simplex  $I$  is a facet of  $P$ .

Notice, that  $V$  does not violate lemma 3.2.

$$v_1 + \dots + v_8 = e_2 + e_5.$$

If  $V$  did contradict lemma 3.2, then the polytope  $P$  could not exist, and **CheckSubset**( $V, \{I\}$ ) should return false.

For simplicity we denote any  $k$ -simplex  $\text{conv}\{v_{i_1}, \dots, v_{i_k}\}$  by  $\{i_1, \dots, i_k\}$ .

Since  $\langle u_I, v_6 \rangle = 0$ , the simplices  $F_1 = \{2, 3, 4, 5, 6\}$  and  $F_2 = \{1, 3, 4, 5, 6\}$  are facets of  $P$  (lemma 2.4).

There are exactly two facets of  $P$  containing the ridge  $\{1, 2, 4, 5\}$ . One of them is  $I$ . Suppose the other one is  $\{1, 2, 4, 5, 9\}$ , where  $v_9$  is some lattice point not in  $V$ ,  $v_9 \in \mathcal{V}(P)$ . Then  $\langle u_I, v_9 \rangle > \langle u_I, v_7 \rangle$  by lemma 2.2.(5) and then  $v_9 \prec v_7$  by the definition of the ordering of lattice points  $\mathbb{Z}^d$ . But then  $V$  is not a presubset of  $\mathcal{V}(P)$ . This is the nice property of the ordering of  $\mathbb{Z}^d$ , and the reason why we chose it as we did. We conclude that  $F_3 = \{1, 2, 4, 5, 7\}$  is a facet of  $P$ , and by similar reasoning  $F_4 = \{1, 2, 3, 5, 8\}$  and  $F_5 = \{1, 2, 3, 4, 8\}$  are facets of  $P$ .

Now, for each of the facets  $F_i$  and every point  $v_j \in V$ , we check if  $\langle u_{F_i}, v_j \rangle = 0$ . If this is the case, then by lemma 2.4  $\text{conv}(\{v_j\} \cup \mathcal{V}(F_i) \setminus \{w\})$  is a facet of  $P$  for every  $w \in \mathcal{V}(F_i)$  where  $\langle u_{F_i}^w, v_j \rangle < 0$ . In this way we get that

$$\{2, 4, 5, 6, 7\} , \{1, 4, 5, 6, 7\} , \{1, 2, 3, 7, 8\} , \{1, 3, 5, 7, 8\}$$

are facets of  $P$ .

We continue in this way, until we cannot deduce any new facet of  $P$ . Every time we find a new facet  $F$  we check that  $v$  is beneath  $F$  (that is  $\langle u_F, v \rangle \leq 1$ ) and that lemma 2.3 holds for any  $v \in V$ . If not, then **CheckSubset**( $V, \{I\}$ ) should return false.

If no contradiction arises, **CheckSubset**( $V, \{I\}$ ) returns the set of deduced facets.

### 5.3 The SFP-algorithm in pseudo-code

Input: A positive integer  $d$ .

Output: A list of special embeddings of smooth Fano  $d$ -polytopes, such that

1. Any smooth Fano  $d$ -polytope is isomorphic to one and only one polytope in the output list.
2. If  $P$  is a smooth Fano  $d$ -polytope in the output list, then  $\mathcal{V}(P) = \text{ord}(P)$ .
3. If  $P_1$  and  $P_2$  are two non-isomorphic smooth Fano  $d$ -polytopes in the output list and  $P_1$  preceeds  $P_2$  in the output list, then  $\text{ord}(P_1) \prec \text{ord}(P_2)$ .

**SFP** ( an integer  $d \geq 1$  )

1. Construct the set  $V = \{e_1, \dots, e_d\}$  and the simplex  $I = \text{conv}V$ .
2. Call the function **AddPoint**( $V, \{I\}$ ).
3. End program.

**AddPoint** ( a subset  $V$  where  $\{e_1, \dots, e_d\} \subseteq V \subseteq \mathcal{W}_d$  , a set of  $(d-1)$ -simplices  $\mathcal{F}$  in  $\mathbb{R}^d$  where  $I \in \mathcal{F}$  )

1. If  $P = \text{conv}(\mathcal{V}(V))$  is a smooth Fano  $d$ -polytope and  $\mathcal{V}(V) = \text{ord}(P)$ , then output  $P$ .
2. Go through every  $v \in \mathcal{W}_d$ ,  $\max \mathcal{V}(V) \prec v$ , in increasing order with respect to the ordering  $\prec$ :
  - (a) If **CheckSubset**( $V \cup \{v\}, \mathcal{F}$ ) returns false, then goto (d). Otherwise let  $\mathcal{F}'$  be the returned set of  $(d-1)$ -simplices.

- (b) If  $V \cup \{v\} \neq \min\{\sigma.(V \cup \{v\}) \mid \sigma \in S_d\}$ , then goto (d).
- (c) Call the function **AddPoint**( $V \cup \{v\}, \mathcal{F}'$ ).
- (d) Let  $v$  be the next element in  $\mathcal{W}_d$  and go back to (a).

3. Return

**CheckSubset** ( a subset  $V$  where  $\{e_1, \dots, e_d\} \subseteq V \subseteq \mathcal{W}_d$ , a set of  $(d-1)$ -simplices  $\mathcal{F}$  in  $\mathbb{R}^d$  where  $I \in \mathcal{F}$  )

1. Let  $\nu = \sum_{v \in V} v$ .
2. If  $\langle u_I, \nu \rangle < 0$ , then return false.
3. If  $\langle u_I^{e_i}, \nu \rangle > 1 + \langle u_I, \nu \rangle$  for some  $i$ , then return false.
4. Let  $\mathcal{F}' = \mathcal{F}$ .
5. For every  $i \in \{1, \dots, d\}$ : If the set  $\{v \in V \mid \langle u_I^{e_i}, v \rangle < 0\}$  is equal to  $\{\max V\}$ , then add the simplex  $\text{conv}(\{\max V\} \cup \mathcal{V}(I) \setminus \{e_i\})$  to  $\mathcal{F}'$ .
6. If there exists  $F \in \mathcal{F}'$  such that  $\mathcal{V}(F)$  is not a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^d$ , then return false.
7. If there exists  $F \in \mathcal{F}'$  and  $v \in V$  such that  $\langle u_F, v \rangle > 1$ , then return false.
8. If there exists  $F \in \mathcal{F}'$ ,  $v \in V$  and  $w \in \mathcal{V}(F)$ , such that

$$\langle u_F^w, v \rangle < \begin{cases} 0 & \langle u_F, v \rangle = 1 \\ -1 & \langle u_F, v \rangle = 0 \\ \langle u_F, v \rangle & \langle u_F, v \rangle < 0 \end{cases}$$

then return false.

9. If there exists  $F \in \mathcal{F}'$ ,  $v \in V$  and  $w \in \mathcal{V}(F)$ , such that  $\langle u_F, v \rangle = 0$  and  $\langle u_F^w, v \rangle = -1$ , then consider the simplex  $F' = \text{conv}(\{v\} \cup \mathcal{V}(F) \setminus \{w\})$ . If  $F' \notin \mathcal{F}'$ , then add  $F'$  to  $\mathcal{F}'$  and go back to step 6.
10. Return  $\mathcal{F}'$ .

## 5.4 Justification of the SFP-algorithm

The following theorems justify the SFP-algorithm.

**Theorem 5.1.** *Let  $P$  be a special embedding of a smooth Fano  $d$ -polytope and  $V$  a presubset of  $\mathcal{V}(P)$ , such that  $\{e_1, \dots, e_d\} \subseteq V$ . Let  $\mathcal{F}$  be a set of facets of  $P$ .*

*Then **CheckSubset**( $V, \mathcal{F}$ ) returns a subset  $\mathcal{F}'$  of the facets of  $P$  and  $\mathcal{F} \subseteq \mathcal{F}'$ .*

*Proof.* By lemma 3.2 the subset  $V$  will pass the tests in step 2 and 3 in **CheckSubset**.

The function **CheckSubset** constructs a set  $\mathcal{F}'$  of  $(d-1)$ -simplices containing the input set  $\mathcal{F}$ . We now wish to prove that every simplex  $F$  in  $\mathcal{F}'$  is a facet of  $P$ : By the assumptions the subset  $\mathcal{F} \subseteq \mathcal{F}'$  consists of facets of  $P$ .

Consider the addition of a simplex  $F_i$ ,  $1 \leq i \leq d$ , in step 5:

$$F_i = \text{conv}(\{\max V\} \cup \mathcal{V}(I) \setminus \{e_i\}).$$

As  $\max V$  is the only element in the set  $\{v \in V \mid \langle u_I^{e_i}, v \rangle < 0\}$  and  $V$  is a presubset of  $\mathcal{V}(P)$ ,  $F_i$  is a facet of  $P$  by lemma 4.2.

Consider the addition of simplices in step 9: If  $F$  is a facet of  $P$ , then by lemma 2.4 the simplex  $\text{conv}(\{v\} \cup \mathcal{V}(F) \setminus \{w\})$  is a facet of  $P$ .

By induction we conclude, that every simplex in  $\mathcal{F}'$  is a facet of  $P$ . Then any simplex  $F \in \mathcal{F}'$  will pass the tests in steps 6–8 (use lemma 2.3 to see that the last test is passed).

This proves the theorem.  $\square$

**Theorem 5.2.** *The SFP-algorithm produces the promised output.*

*Proof.* Let  $P$  be a smooth Fano  $d$ -polytope. Clearly,  $P$  is isomorphic to at most one polytope in the output list.

Let  $Q$  be a special embedding of  $P$  such that  $\mathcal{V}(Q) = \text{ord}(P)$ . We need to show that  $Q$  is in the output list. Let  $\mathcal{V}(Q) = \{e_1, \dots, e_d, q_1, \dots, q_k\}$ , where  $q_1 \prec \dots \prec q_k$ , and let  $V_i = \{e_1, \dots, e_d, q_1, \dots, q_i\}$  for every  $1 \leq i \leq k$ .

Certainly the function **AddPoint** has been called with input  $\{e_1, \dots, e_d\}$  and  $\{I\}$ .

By theorem 5.1 the function call **CheckSubset**( $V_1, \{I\}$ ) returns a set  $\mathcal{F}_1$  of  $(d-1)$ -simplices which are facets of  $Q$ ,  $I \subset \mathcal{F}_1$ . By lemma 4.5 the set  $V_1$  passes the test in 2b in **AddPoint**. Then **AddPoint** is called recursively with input  $V_1$  and  $\mathcal{F}_1$ .

The call **CheckSubset**( $V_1, \mathcal{F}_1$ ) returns a subset  $\mathcal{F}_2$  of facets of  $Q$ , and the set  $V_2$  passes the test in 2b in **AddPoint**. So the call **AddPoint**( $V_2, \mathcal{F}_2$ ) is made.

Proceed in this way to see that the call **AddPoint**( $V_k, \mathcal{F}_k$ ) is made, and then the polytope  $Q = \text{conv} V_k$  is outputted in step 1 in **AddPoint**.  $\square$

## 6 Classification results and where to get them

A modified version of the SFP-algorithm has been implemented in C++, and used to classify smooth Fano  $d$ -polytopes for  $d \leq 7$ . On an average home computer our program needs less than one day (january 2007) to construct the classification list of smooth Fano 7-polytopes. These lists can be downloaded from the authors homepage: <http://home.imf.au.dk/oebro>

An advantage of the SFP-algorithm is that it requires almost no memory: When the algorithm has found a smooth Fano  $d$ -polytope  $P$ , it needs not consult the output list to decide whether to output the polytope  $P$  or not. The construction guarantees that  $\mathcal{V}(P) = \min\{\sigma.\mathcal{V}(P) \mid \sigma \in S_d\}$  and it remains to check if  $\mathcal{V}(P) = \text{ord}(P)$ . Thus there is no need of storing the output list.

The table below shows the number of isomorphism classes of smooth Fano  $d$ -polytopes with  $n$  vertices.

$n$	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$
1							
2	1						
3		1					
4		2	1				
5		1	4	1			
6		1	7	9	1		
7			4	28	15	1	
8			2	47	91	26	1
9				27	268	257	40
10				10	312	1318	643
11				1	137	2807	5347
12				1	35	2204	19516
13					5	771	26312
14					2	186	14758
15						39	4362
16						11	1013
17						1	214
18						1	43
19							5
20							2
Total	1	5	18	124	866	7622	72256

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